respond instantly to vehicle motions. Consequently the $\epsilon^*(t)$ signal continues to increase at a constant rate until t_1. At instant t_1, a new pulse width is calculated which will be longer than that calculated at t_0, and this now becomes the thruster command. Assuming that τ_0 is longer than T, this simply means that at t_1 the thruster command is continued. Because of the thruster delay, the gyro dynamics, and the sample and hold operation, the $\epsilon^*(t)$ signal will remain unaffected by the vehicle acceleration caused by the thruster for some time after t_1. The thruster will actually come on at time $t_0+\tau_D$, which would be sometime around t_1, if T and τ_D are approximately the same. Here we show $\epsilon^*(t)$ increasing at its constant rate until t_2, the sample instant at which the effect of the thruster is first sensed at the pulse width modulator. $\epsilon^*(t_2)$ is even larger than $\epsilon^*(t_1)$, however, so the pulse width calculated at t_2 will be longer than that at t_1, and the thrust command will be continued. $\epsilon^*(t)$ will therefore remain above the dead-zone level for several sample periods. Here it drops below W for the first time at t_{10}, so at t_{10} the thrust command is set to zero. At some later instant, shown at t_{11} here, $\epsilon^*(t)$ will be in the dead-zone and decreasing at a constant rate.

The thruster pulse duration during this reversal in $\epsilon^*(t)$ is much longer than the minimum thruster on time because of the time delays in the control loop. Therefore, the vehicle rate during the limit cycle oscillation is higher than the minimum which is physically possible, so the limit cycle is less efficient than is physically possible.

A new control law, designed to overcome the effects of the time lag and improve the limit cycle performance, has been devised. This scheme utilizes the logical capabilities of the digital computer in addition to its computational abilities, and is called the inhibitor model of control.[1]

The basic inhibitor mode works during the limit cycle in the following way. Referring to Fig. 3, at t_0 the pulse width is calculated and the thruster command issued in the same manner as in the conventional scheme described. In the case shown here, this pulse width will be slightly larger than the minimum τ_0 because $\epsilon^*(t_0)$ happens to be slightly greater than W. At the time the pulse command is issued the computer switches into the "inhibit" mode, which simply means that $\epsilon^*(t)$ will be ignored (assumed to be zero) for a given period of time—on the order of a few cycle periods. The pulse which is commanded at t_0 will be emitted, the vehicle will decelerate and reverse direction, and the gyro will react and cause $\epsilon^*(t)$ to reverse direction and even drop below the threshold level W during the period of inhibition. When the inhibit period is completed, the computer reverts to its alert mode and will command a corrective pulse the next time ϵ^* exceeds the threshold level.

To investigate the feasibility of the inhibitor scheme, a simulation of the system shown in Fig. 1 was established. For this simulation, the parameter values shown in Table 1 were used. T_I is the inhibit period. In each case reported here, a two-pulse limit cycle, symmetrical in ω, was established. The results of these tests are summarized in Fig. 4 where the pulse duration which resulted in each case is plotted against the K_R gain used. In the conventional case the limit cycle was unstable (or did not exist) for $K_R<0.5$.

Table 1 Parameter values used in simulation

τ_D	$=1$
A	$=0.02$
J	$=1$
ω_n	$=1.2$
ζ	$=0.7$
T	$=1$
K_P	$=0.2$
K_R	= variable, see Fig. 4
W	$=1$
τ_0	$=1.5$
S	$=1$
T_I	= 0 for conventional control law
T_i	= 5 for inhibitor mode

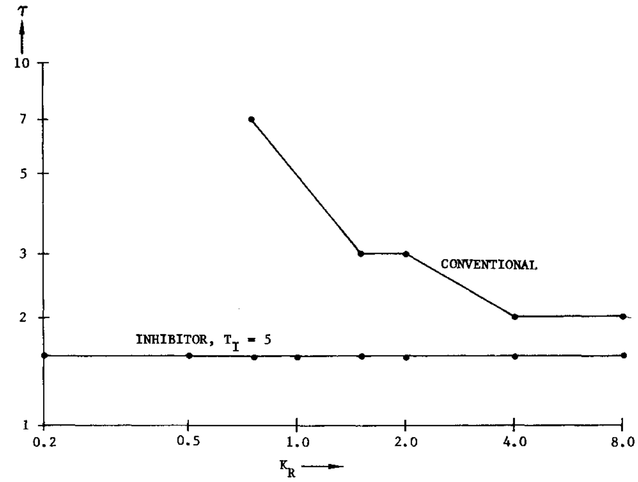

Fig. 4 Pulse-width vs K_R for $K_P=0.2$.

For a symmetrical limit cycle, the important performance parameters can be established from τ. In most attitude control systems the positional accuracy is essentially determined by the dead-zone width W, so is independent of τ. The long-term fuel expenditure is proportional to τ^2, so if this is taken as the performance measure, we see that the inhibitor scheme, in the system tested here, gave an improvement in performance over the conventional control law ranging from 20.4 to 1.7, depending on K_R.

There are many possible variations of the basic inhibitor scheme which is described here. Stability of two pulse limit cycles and the existence of higher order limit cycles are important considerations in any extensions of the inhibitor idea. In the presence of disturbance torques, a situation not investigated here, it would probably be necessary to provide for a variable inhibit period T_I to prevent excessive excursions of ω.

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Velocity of Bodies Powered by Rapidly Discharged Cold-Gas Thrusters

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Nomenclature

c_i	= initial sound speed	(m/sec)
I_s	= specific impulse	(m/sec)
k	= specific-heats parameter, $k=2/(\gamma-1)$	
m_f	= final vehicle mass	(kg)

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m_g = initial propellant (gas) mass, $m_g = m_i - m_f$ (kg)
m_i = initial vehicle mass, including propellant (gas) (kg)
S = nozzle throat area (m^2)
t = time (sec)
V = tank volume (m^3)
$\mathcal{V}_S$ = normalized velocity, $\mathcal{V}_S \equiv v/v_{t=\infty}$
v = vehicle translational velocity (m/sec)
β = specific-heats factor, $\beta = [(\gamma-1)/2][(\gamma+1)/2]^{-\zeta/2}$
γ = ratio of specific heats
ζ = specific-heats function, $\zeta = (\gamma+1)/(\gamma-1)$
θ = specific-heats factor, $\theta = [2/(\gamma+1)][2/(\gamma-1)]^{1/2}$
λ = vehicle mass ratio, $\lambda = m_f/m_i$
τ = normalized time, $\tau = 1 - 1/(1+\psi t)$
ψ = configuration parameter, $\psi = \beta S c_i / V$ (1/sec)

Introduction

INERT propellant systems such as cold-gas thrusters are used extensively as vehicle attitude control devices, and technology is fairly advanced.[1] A less common mode of operation involves the use of this type of engine to effect the vehicle translational motion. The following discussion concerns the ideal velocity imparted to a missile as a result of discharging the chemically nonreacting fluid from a thermally insulated tank.

As might be expected, the velocity equation that evolves in the absence of heat addition is fundamentally different from the well-known formula that pertains to rocket flight in a gravitationless vacuum. According to the latter expression, the magnitude of the resultant velocity, $v = v_e \ln(1/\lambda)$, where v_e, λ denote the effective exhaust velocity and the vehicle mass ratio, respectively,[2] approaches infinity if the propellant mass fraction becomes large ($\lambda \to 0$), even though the exhaust velocity may be small. The limiting velocity derived here remains finite. Its magnitude is determined by the molecular degrees of freedom and the initial sound speed in the compressed gas.

Equation of Motion

In the absence of forces other than the reaction due to releasing the inert working fluid, the net force F applied to the body is

$$F = pSC_T \tag{1}$$

where $p = p(t)$ is the gas pressure in the tank. Generally, the thrust coefficient C_T is a function of the tank pressure,[3] which is time-dependent. In ideal flows with fixed exhaust geometry, however, the coefficient becomes invariant if the ambient pressure is zero. Its magnitude then depends only on the nozzle exit-area ratio and the ratio of specific heats. The maximum value occurs in a vacuum, with complete expansion of the gas, which implies an infinite area-ratio nozzle. In this case the coefficient reduces to a function involving only the specific-heat ratio:

$$C_T = \gamma\{[(\gamma-1)/2][(\gamma+1)/2]^{(\gamma+1)/(\gamma-1)}\}^{-1/2} \tag{2}$$

If the thermodynamic properties of the fluid inside the vessel maintain a uniform spatial distribution during the discharge process, the time-dependent gas dynamic equations can be analytically integrated to give the pressure and temperature histories of a perfect gas with no thermal radiation, provided that the flow is assumed to be one-dimensional, quasi-steady and isentropic. In that domain of validity, the variations of average pressure and temperature in an insulated tank of fixed geometry (out of which issues a choked flow) or, conversely, in a tank discharged at a sufficiently rapid rate to preclude significant heat transfer, are[4,5]

$$p = p_i(1+\psi t)^{-(k+2)} \tag{3}$$

and

$$T = T_i(1+\psi t)^{-2} \tag{4}$$

respectively, where ψ, k are known constants and subscript i denotes initial values.

By combining Eqs. (3) and (4) with the equation of state, the mass of the transferred propellant is obtained. Consequently, the instantaneous mass of the system being accelerated becomes

$$m = \lambda m_i + (1-\lambda)(1+\psi t)^{-k} m_i \tag{5}$$

where $\lambda = m_f/m_i$. Then from Eqs. (1-3) and (5), the governing expression for the translational velocity of the variable-mass body takes the form

$$v = a\int_{t_1}^{t_2} dt/[(1+\psi t)^2 + b(1+\psi t)^{k+2}] \tag{6}$$

in which $a = p_i S C_T/(m_i - \lambda m_i)$ and $b = \lambda/(1-\lambda)$.

Equation (6) may be simplified by introducing a normalized time, $\tau \equiv 1-(1+\psi t)^{-1}$, and changing variables in accordance with the relation $\tau = 1 - b^{1/k}\xi$. When $t_1 = 0$ and $t_2 = t$, the final result is

$$v = \zeta I_s(\tau - \mathcal{J}_\tau) \tag{7}$$

where the definite integral

$$\mathcal{J}_\tau = b^{1/k}\int_{\xi_1}^{\xi_2} d\xi/(1+\xi^k)$$

has integration limits of $\xi_1 = (1/b)^{1/k}(1-\tau)$, $\xi_2 = (1/b)^{1/k}$. The premultipliers of Eq. (7) are $\zeta = (\gamma+1)/(\gamma-1)$ and

$$I_s = (1/m_g)\int_0^\infty F dt = \theta c_i$$

where

$$\theta = [2/(\gamma+1)][2/(\gamma-1)]^{1/2}$$

Limiting Velocity

The limiting velocity, i.e., the velocity of a vehicle composed entirely of an inert propulsive gas, may be derived from Eq. (7). For that purpose, it may be shown that the integral $\int d\xi/(1+\xi^k)$ never exceeds finite bounds in any possible condition, including $\lambda = 0$ ($1 < k < \infty$). Hence, at the boundary where $\lambda = 0$, $\mathcal{J}_\tau = 0$ when $\tau = 1$ and the corresponding velocity becomes $v_{lim} = \zeta I_s$. Consequently, expressed in terms of the initial sound speed, the maximum possible velocity attainable from a thermally insulated vessel is $[2/(\gamma-1)]^{3/2} c_i$, or

$$v_{lim} = k^{3/2} c_i \tag{8}$$

The limiting velocity is unbounded only if $c_i = \infty$. It is noted that Eq. (8) can also be obtained from the equation of motion, Eq. (6), by direct integration for the specific case of $\lambda = 0$.

Velocity History

The integral function that occurs in Eq. (7) may be evaluated numerically in its present form to determine the velocity history, or a general solution can be constructed by replacing the integrand with a geometric series and analytically integrating each term separately. With the latter procedure, companion equations evolve. If $\xi^{2k} < 1$,

$$\mathcal{J}_\tau = b^{1/k}\int \sum_{j=0}^{\infty} (-1)^j \xi^{jk} d\xi \tag{9a}$$

and if $\xi^{2k} > 1$,

$$\mathcal{I}_\tau = b^{1/k} \int \sum_{j=1}^{\infty} (-1)^j \xi^{-jk} d\xi \qquad (9b)$$

When $\lambda > 1/2$, Eq. (9a) suffices, and the limits of integration remain the same as those of Eq. (7). When $\lambda < 1/2$, both equations are required; the independent variable ranges from $(1/b)^{1/k}(1-\tau) < 1$ to 1 in Eq. (9a), and from 1 to $(1/b)^{1/k}$ in Eq. (9b).

For the special case in which k may be specified as a positive integer, the integral function has a finite-series solution

$$\mathcal{I}_\tau = (b^{1/k}/k)\{s \ln(\xi+1) - 2\sum_{\mu=0}^{r-1} [P_\mu \cos(\eta\pi/k) - Q_\mu \sin(\eta\pi/k)]\} \qquad (10)$$

with $k = 2r - s$, where $s = 0$ or 1. The trigonometric coefficients are $P_\mu = (1/2)\ln[\xi^2 - 2\xi\cos(\eta\pi/k) + 1]$ and $Q_\mu = \arctan[\xi \csc(\eta\pi/k) - \cot(\eta\pi/k)]$, where $\eta = 2\mu + 1$. With $\mathcal{I}_\tau$ determined by Eqs. (9) or (10), the ideal velocity history of any given system may now be calculated using Eq. (7).

It is noted that according to the kinetic theory of gases for an elementary molecular model, the number (n) of degrees of freedom of the molecule is related to the ratio of specific heats by the formula $n = 2/(\gamma - 1)$. Since this expression is identical to the present generalized definition of k, the integer k in Eq. (10) becomes equivalent to the internal degrees of freedom.

Monatomic Gases

The simplest molecular model pertains to the monatomic gases and, for this particular group of fluids, Eq. (10) is expanded and combined with Eq. (7) to describe in detail the resultant velocity of rapidly discharged tanks.

If only three degrees of freedom exist, $\gamma = 5/3$, and consequently, the equation that expresses the approximate velocity of isentropically discharged thrusters reduces to

$$\mathcal{V}_S = (\tau - A_\tau)/(1 - A_1) \qquad (11)$$

where $\mathcal{V}_S \equiv v/v_{t=\infty}$ denotes the normalized velocity of the constant-entropy system and $A_1 \equiv A_{\tau=1}$, where $3\alpha A_\tau = \ln(f_1 f_2) + 3^{1/2} \arctan(f_3)$. The dimensionless f-functions are $f_1 = (\alpha+1)/[\alpha(1-\tau)+1]$, $f_2{}^2 = [\alpha^2(1-\tau)^2 - \alpha(1-\tau) + 1]/(\alpha^2 - \alpha + 1)$, and $f_3 = 3^{1/2}\tau/[(2\alpha - 1)(1-\tau) + (2/\alpha) - 1]$, where $\alpha^3 = (1-\lambda)/\lambda$.

Maximum velocity occurs at $\tau = 1$, and amounts to $v_{t=\infty} = \zeta I_s(1 - A_1)$. Written in terms of the initial sound speed,

$$v_{t=\infty} = 3^{3/2}(1 - A_1)c_i \qquad (12)$$

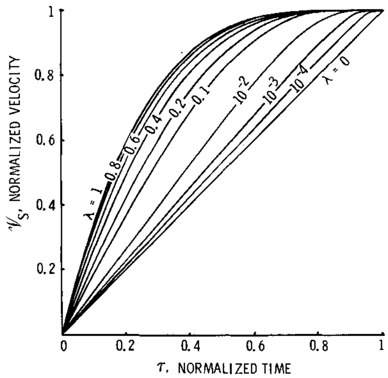

Fig. 1 Velocity history showing effect of mass ratio (λ) for monatomic gases.

It may be shown that $A_1 \to 1$ as $\lambda \to 1$ and, from Eq. (12), the corresponding velocity duly reduces to zero. At the other extreme, where $\lambda = 0$, the dimensionless parameter A_1 becomes equal to zero, and hence the limiting velocity becomes $v_{\lim} = 3^{3/2}c_i$, which is identical to the result obtained from the general expression, Eq. (8). Between these extremes, it is found that if the propellant mass fraction is not large, the gas helium, for example, when taken to be initially at room temperature, produces a terminal velocity that is approximately two-thirds the vacuum velocity of conventional solid-fuel chemical rocket engines of the same mass fraction. Argon produces about one-fifth of the chemical rocket velocity.

Intermediate values of the dimensionless velocity parameter $(1 - A_1)$, which occurs in Eqs. (11) and (12), have also been computed. They are as follows for mass fractions (λ) varying from zero to one in one-tenth increments: 1, 0.4719, 0.3524, 0.2734, 0.2133, 0.1644, 0.1229, 0.08685, 0.05488, 0.02614, 0, respectively.

The velocity determined from Eq. (11) is shown parametrically in Fig. 1, with the gas mass fraction ranging over the complete spectrum. From these results, the velocity history of monatomic propellant systems with infinite area-ratio nozzles may be readily calculated.

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